AFOSR-TN-58-23

ASTIA Document No. AD 148 062

CALIFORNIA INSTITUTE OF TECHNOLOGY

Antenna Laboratory

Technical Report No. 14

ON THE HIGH-FREQUENCY OSCILLATIONS OF THE ELECTRONIC PLASMA

by

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Research supported by

the U. S. Air Force Office of Scientific Research

January 1958

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Introduction:

For the purpose of this note an electronic plasma is defined as a gas of classical, non-relativistic* electrons immersed in a constant charge-neutralizing background. The plasma is assumed to be spatially limitless and free of externally applied fields.

An exact analysis of the general oscillatory behavior of the plasma is forbiddingly difficult because it requires a detailed knowledge of the collision mechanism and ultimately leads to an intractable integro-differential equation. However, there are two extreme cases that are simple enough to be handled mathematically.

One of these limiting cases occurs when the collisions are so frequent that the electronic distribution is Maxwellian in every volume element and local equilibrium is established. Then the behavior of the plasma is determined by macroscopic hydrodynamical equations which lead to the dispersion relation**

$$\omega^2 = \omega_p^2 + \frac{5}{3}\left(\frac{\varkappa T}{m}\right) k^2 \qquad \omega^2 = \omega_p^2 + (5/3)(\varkappa T/m)k^2$$

where ω_p is the plasma frequency*** given by

$$\omega_p^2 = \frac{ne^2}{m\epsilon_o} \qquad \omega_p^2 = ne^2/m\epsilon_0$$

with T denoting the equilibrium temperature, $\varkappa$ Boltzmann's constant, k the wave number, m and e the electronic mass and charge respectively, and ϵ_o the dielectric constant of free space (M.K.S. system). This dispersion relation does not agree with the dispersion relation

*For relativistic considerations see P. C. Clemmow and A. J. Wilson, Proc. Royal Soc. 237, 117 (1956).

**Gross, E. P., Phys. Rev. 82, 232 (1951).

***Tonks, L. and I. Langmuir, Phys. Rev. 33, 195 (1929); Lord Rayleigh, Phil. Mag. 11, 117 (1906).

†Based on a lecture presented at the Highes Aircraft Company.

derived by the Thomsons*, by Bailey**, by Borgnis***, and others. The reasons for this discrepancy have been reported by Van Kampen.**** In the other limiting case the collisions of the electrons with the ions and with each other are negligible and the collision term of Boltzmann's equation can be set equal to zero. This state is physically approximated when the frequency of oscillation is sufficiently high. Under special circumstances the dispersion relation in this case is approximately given by*****

$$\omega^2 = \omega_p{}^2 + 3(\frac{\varkappa T}{m})\, k^2 \quad . \qquad \omega^2 = \omega_p^2 + 3(\varkappa T/m)k^2$$

In this lecture we shall critically examine the theory of the high-frequency case, placing in evidence the tacit assumptions and hypotheses upon which the theory is based.

Formulation of the Problem.

Since the charge-neutralizing background of the plasma is constant, we need to know the distribution function of only the electrons. The electronic distribution function $F(\underline{r},\underline{v},t)$ which is taken to be a sufficiently well-behaved function of position $\underline{r}$, velocity $\underline{v}$, and time t, must satisfy Boltzmann's equation

$$\frac{\partial F}{\partial t} + \underline{v} \cdot \nabla F - \frac{e}{m} (\underline{E} + \underline{v} \times B) \cdot \nabla_v F = C \tag{1}$$

where ∇ and ∇_v are gradient operators in coordinate space and velocity space respectively. Since the effect of collisions is unessential at high frequencies, the collision term C may be set equal to zero; and since only irrotational oscillations are to be considered, the macroscopic magnetic field $\underline{B}$ vanishes and the macroscopic electric field $\underline{E}$ is derivable from a scalar potential $\phi(\underline{r},t)$ alone, i.e.,

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* Thomson, J.J. and G. P. Thomson, Conduction of Electricity through Gases, third edition, Cambridge, 1933, p.353.

**Bailey, V.A., Phys. Rev. 78, 428 (195).

***Borgnis, F. E., Helvetica Physica Acta, 20, 207 (1947).

****Van Kampen, N.G., Physica, Vol.33, No.7, July 1957.

*****Vlasov, A.A., J. Exp. Theor. Phys. USSR, 8, 291 (1938); J.Phys. USSR, 9, 25 (1945). Landau, L., J. Phys. USSR, 10, 25 (1946).

$$\underline{E} = -\nabla\phi \ . \tag{2}$$

The scalar potential is macroscopic* and thus does not exhibit the rapid variations of a Lorentzian microscopic potential. In this scheme of partition the effect of the rapid variations are lumped into C along with the effects of the short-range forces. Thus Boltzmann's equation reduces to

$$\frac{\partial F}{\partial t} + \underline{v} \cdot \nabla F + \frac{e}{m} \nabla \phi \cdot \nabla_v F = 0 \ . \tag{3}$$

In the equilibrium state charge neutrality prevails and the positive charge of the background cancels the negative charge of the electrons. Hence, if $n_o f_o(\underline{v})$ is the equilibrium distribution of the electrons where n_o is a constant and $f_o(\underline{v})$ is normalized to unity, the positive charge density of the neutralizing background is the constant

$$e\, n_o \int_{-\infty}^{\infty} f_o \, d\tau = e\, n_o \tag{4}$$

where $d\tau = dv_x\, dv_y\, dv_z$. Hence the resulting charge density at point $\underline{r}$ and time t is given by

$$\rho(\underline{r},t) = e\, n_o - e \int_{-\infty}^{\infty} F d\tau \ . \tag{5}$$

For the sake of linearity it is assumed that the non-equilibrium distribution F of the electrons differs from the equilibrium distribution $n_o f_o$ by only a small amount f . This restriction confines the subsequent analysis to small signals. Accordingly we have

$$F(\underline{r},\underline{v},t) = n_o f_o(\underline{v}) + f(\underline{r},\underline{v},t) \tag{6}$$

where $f \ll n_o f_o$ and $\nabla_v f \ll n_o \nabla_v f_o$. Under this assumption the reduced Boltzmann's equation (3) and expression (5) for the charge density $\rho(\underline{r},t)$ become respectively

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* Granowski, W.L., Der Elektrische Strom im Gas, Berlin (1955).

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \nabla f + \frac{n_o e}{m} \nabla \phi \cdot \nabla_v f_o = 0 \;, \tag{7}$$

and

$$\rho(\underline{r},t) = -e \int_{-\infty}^{\infty} f \, d\tau \;. \tag{8}$$

The scalar potential is macroscopic and is therefore related to the charge density by Poisson's equation

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_o} = \frac{e}{\epsilon_o} \int_{-\infty}^{\infty} f \, d\tau \tag{9}$$

or equivalently by

$$\phi(\underline{r},t) = -\frac{e}{4\pi\epsilon_o} \int_{-\infty}^{\infty} \frac{f(\underline{r},\underline{v}',t)}{|r - r'|} \, d\tau' \, d\Omega' \tag{10}$$

where $d\Omega = dx\, dy\, dz$. Substituting expression (10) into equation (7) we see that $f(\underline{r},\underline{v},t)$ must satisfy the integro-differential equation

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \nabla f + \frac{\omega_p^2}{4\pi} \nabla_v f_o \cdot \int_{-\infty}^{\infty} \frac{\underline{r} - \underline{r}'}{|\underline{r} - \underline{r}'|^3} f(\underline{r}',\underline{v}',t) d\tau' \, d\Omega' = 0 \;. \tag{11}$$

This is the basic equation for the small-signal theory of high-frequency oscillations and was first derived by Vlasov.*

Vlasov's Theory.

To find a wave solution of the integro-differential equation (11), Vlasov assumes that the distribution function f is of the form

$$f(\underline{r},\underline{v},t) = g_k(\underline{v}) \; e^{i\underline{k}\cdot\underline{r}} \; e^{-i\omega t} \tag{12}$$

where $\underline{k}$ is a vector wave-number and ω an angular frequency. Substitution of this expression into the integro-differential equation (11) leads to

*Vlasov, A. A., Jour. Exp. and Theor. Phys., 8 291 (1938); Jour. of Phys. USSR, 9, 25 (1945).

$$g_k(\underline{v}) = \frac{\omega_p^2}{k^2} \frac{(\underline{k} \cdot \nabla_v f_o)}{\underline{k} \cdot \underline{v} - \omega} \int_{-\infty}^{\infty} g_k(\underline{v}') \, d\tau' \tag{13}$$

which in turn yields (after an integration with respect to v_x, v_y, v_z) Vlasov's dispersion equation

$$\frac{\omega_p^2}{k^2} \int_{-\infty}^{\infty} \frac{\underline{\underline{k}} \cdot \nabla_v f_o}{\underline{k} \cdot \underline{v} - \omega} \, d\tau = 1 \, . \tag{14}$$

In view of the singularity in the integrand, it is necessary to specify how the pole is to be dealt with. Vlasov takes the Cauchy principle value. The question of the singularity may be avoided by assuming that $f_o(\underline{v})$ disappears in some range of $\underline{v}$ containing the zeros of the denominator. The use of this rather artificial assumption may be justified on the grounds that it simplifies the calculations and at least coarsely approximates a distribution with a small tail beyond v_{max}. With the aid of this hypothesis, equation (14) when integrated by parts, yields

$$\omega_p^2 \int_{-\infty}^{\infty} \frac{f_o(\underline{v})}{(\underline{k} \cdot \underline{v} - \omega)^2} \, d\tau = 1 \, , \tag{15}$$

which agrees with the determinental equations differently derived by Bohm and Gross* and by Pines and Bohm**. Specifically, if $f_o(\underline{v}) = 0$ for $|v| > v_{max}$ and if $\omega > kv_{max}$, the integrand may be expanded in powers of $\underline{k} \cdot \underline{v}/\omega$. Thus

$$1 = \frac{\omega_p^2}{\omega^2} \int_{-\infty}^{\infty} \left[1 + 2 \frac{\underline{\underline{k}} \cdot \underline{\underline{v}}}{\omega} + 3 \frac{(\underline{k} \cdot \underline{v})^2}{\omega^2} + \cdots \right] f_o(\underline{v}) \, d\tau \; . \tag{16}$$

Recalling that

$$\int_{-\infty}^{\infty} f_o(\underline{v}) \, d\tau = 1 \quad , \quad \int_{-\infty}^{\infty} (\underline{k} \cdot \underline{v}) \, f_o(\underline{v}) \, d\tau = k \langle \underline{v} \rangle_o \quad ,$$

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* Bohm, D. and E.P. Gross, Phys. Rev. 75, 1851 (1949).
**Pines, D. and D. Bohm, Phys. Rev. 85, 338 (1952).

$$\int_{-\infty}^{\infty} (\underline{k} \cdot \underline{v})^2 \, f_o(\underline{v}) \, d\tau \;=\; k^2 \langle \underline{v}^2 \rangle_o \quad , \text{ etc.}$$

and noting that in the case of an isotropic distribution $f_o(\underline{v}) = f_o(v)$, $\langle \underline{v} \rangle_o = 0$, $\langle v^2 \rangle_o = \frac{\mathcal{K}T}{m}$, we see that if f_o is isotropic, equation (16) yields

$$\omega^2 = \omega_p^2 + 3\frac{k^2 \mathcal{K}T}{m} .$$

$$1 \;=\; \frac{\omega_p^2}{\omega^2} \left(1 + 0 + 3\,\frac{k^2}{\omega^2}\left(\frac{\mathcal{K}T}{m}\right) + \cdots \right) \quad . \qquad (17)$$

Keeping terms up to the second order in ω , we get

$$\omega^2 \;=\; \omega_p^2 \,(1 + 3\,\rho^2\,k^2) \qquad (18)$$

where $\rho^2 = \frac{\mathcal{K}T}{ne^2}\,\epsilon_o$, ρ being the electronic Debye-Huckel radius.

Thus the cut-off hypothesis leads to the dispersion relation (18) when $\omega/k > v_{max}$. However, when $\omega/k < v_{max}$ it is clear that no dispersion relation can exist because for any real value of k a continuous band of real values for ω are possible, extending from $-kv_{max}$ to $+kv_{max}$. When the dispersion relation is not cut-off and the Cauchy principle value is taken, the dispersion relation (18) is still valid for large values of ω/k .

Landau's Theory.

According to Landau the "Ansatz" solution found by Vlasov is not complete on two counts. One of these is the divergent nature of the integral of the dispersion equation (14); the other is that solutions of the form $e^{i\underline{k}\cdot\underline{r}}\, e^{-i\omega t}$ give only a 3-fold infinity of solutions, a single infinity for each of three components, k_x, k_y, k_z, whereas there must actually be a 6-fold infinity of solutions of the form $g_k(\underline{v})\, e^{i\underline{k}\cdot\underline{r}}\, e^{-i\omega t}$, one for each of six components $v_x, v_y, v_z, k_x, k_y, k_z$. To obviate these shortcomings Landau solved the problem as an initial-value problem. He assumed that the equilibrium distribution $n_o f_o(\underline{v})$ was equal to the Maxwell distribution $n_o f_o(v)$, and from a knowledge of $f(\underline{r},\underline{v},0)$ he computed $f(\underline{r},\underline{v},t)$.

In Landau's treatment $f(\underline{r},\underline{v},t)$ and $\phi(\underline{r},t)$ are written as Fourier integrals

$$f(\underline{r},\underline{v},t) = \int_{-\infty}^{\infty} f_k(\underline{v},t)\, e^{i\underline{k}\cdot\underline{r}}\, dk_x\, dk_y\, dk_z \tag{19}$$

$$\phi(\underline{r},t) = \int_{-\infty}^{\infty} \phi_k(t)\, e^{i\underline{k}\cdot\underline{r}}\, dk_x\, dk_y\, dk_z \quad . \tag{20}$$

When these expressions are substituted in Boltzmann's equation (7) and Poisson's equation (9), and when it is assumed that $\underline{k}$ points in the x-direction, it is seen that the Fourier components f_k and ϕ_k satisfy

$$\frac{\partial f_k}{\partial t} + ik\, v_x\, f_k + \frac{ik\, n_o e}{m}\, \phi_k \frac{\partial f_o}{\partial v_x} = 0 \tag{21}$$

$$k^2\, \phi_k = -\frac{e}{\epsilon_o} \int_{-\infty}^{\infty} f_k\, d\tau \quad . \tag{22}$$

In preparation for taking the Laplace transform of these equations we let $\overline{f_k}(\underline{v})$ denote the Laplace transform of $f_k(\underline{v})$ with respect to time, i.e.

$$\overline{f_k}(\underline{v}) = \int_0^{\infty} f_k(\underline{v},t)\, e^{-pt}\, dt \tag{23}$$

and

$$f_k(\underline{v},t) = \frac{1}{2\pi i} \int_{-i\infty+\sigma}^{i\infty+\sigma} \overline{f_k}(\underline{v})\, e^{pt}\, dp \, , \tag{24}$$

where the integration is along the straight line $Re(p) = \sigma > 0$ in the complex p-plane. Similarly we denote by $\overline{\phi_k}$ the Laplace transform of $\phi_k(t)$ with respect to time. With this notation the Laplace transforms of equations (21) and (22) are

$$(p + ikv_x)\, \overline{f_k}(\underline{v}) + \frac{ik\, n_o e}{m}\, \overline{\phi_k}\, \frac{\partial f_o}{\partial v_x} = f_k(\underline{v},0) \tag{25}$$

$$k^2 \, \overline{\phi_k} = - \frac{e}{\epsilon_o} \int_{-\infty}^{\infty} \overline{f_k}(\underline{v}) \, d\tau \quad . \tag{26}$$

It follows algebraically from these equations that

$$\overline{\phi_k} = - \frac{1}{k^2} \frac{e}{\epsilon_o} \frac{\displaystyle\int_{-\infty}^{\infty} \frac{f_k(\underline{v},0)}{p + ik\, v_x} \, d\tau}{1 - \dfrac{i\, \omega_p^2}{k} \displaystyle\int_{-\infty}^{\infty} \frac{\partial f_o / \partial v_x}{p + ik\, v_x} \, d\tau} \quad . \tag{27}$$

The unessential integrations with respect to v_y and v_z can be carried out. Since $f_o(v)$ is a Maxwellian distribution, we see that

$$\int_{-\infty}^{\infty} f_o(v) \, dv_y \, dv_z = f_o(u) \tag{28}$$

where $u \equiv v_x$. Moreover, we define $g_k(u)$ by

$$\int_{-\infty}^{\infty} f_k(\underline{v},0) \;\; dv_y \, dv_z = g_k(u) \quad . \tag{29}$$

Thus after these integrations have been performed equation (27) becomes

$$\overline{\phi_k} = - \frac{1}{k^2} \cdot \frac{e}{\epsilon_o} \; \frac{\displaystyle\int_{-\infty}^{\infty} \frac{g_k(u)}{p + iku} \, du}{1 - i \dfrac{\omega_p^2}{k} \displaystyle\int_{-\infty}^{\infty} \frac{d\, f_o(u)}{du} \, \frac{du}{p + ik\, u}} \quad . \tag{30}$$

This equation is meaningful only in the right-half of the p-plane, $\mathrm{Re}(p) > 0$. To make it meaningful in the left-half of the p-plane, the path of integration must be changed. It is assumed that $g_k(u)$ is an entire function of u , and since $f_o(u)$ is the Maxwellian distribution $f_o'(u)$ is an entire function also. Consequently $g_k(u)$ and $f_o'(u)$ are free of singularities in the finite part of the complex u-plane. It follows that the integrands $g_k(u)/(p + iku)$ and $f'_o(u)/p + iku)$ have poles of order unity at $u = ip/k$. When $Re(p) > 0$ the pole lies in the upper half of the u-plane and the integration from $-\infty$ to ∞ along the real axis of the u-plane is meaningful. However, when $Re(p) < 0$ the pole lies in the lower half of the u-plane and the path of integration must be displaced far enough into the lower half of the u-plane so that $u = ip/k$ would lie above it. Thus we see that if we want equation (30) to be meaningful for $Re(p) < 0$, we must carry out the integrations along a contour C of the u-plane (Figure 1). That is,

$$\overline{\phi_k} = - \frac{1}{k^2} \frac{e}{\epsilon_o} \frac{\displaystyle\int_C \frac{g_k(u)\, du}{p + ik\, u}}{1 - i \frac{\omega_p^2}{k} \displaystyle\int_C \frac{f'_o(u)\, du}{p + iku}} \tag{31}$$

is an analytic function of p for $Re(p) < 0$, whereas expression (30) is an analytic function of p for $Re(p) > 0$. Expression (31) is the analytic continuation of expression (30). The poles of expression (31) lie in the left-half of the p-plane and are determined by

$$\frac{i\omega_p^2}{k} \int_C \frac{f'_o(u)\, du}{p + iku} = 1 \quad . \tag{32}$$

We note that this equation is the same as Vlasov's dispersion equation (14) except for the path of integration.

From a knowledge of $\overline{\phi_k}$ we can find $\phi_k(t)$ by the inversion formula

$$\phi_k(t) = \frac{1}{2\pi i} \int_{-i\infty + \sigma}^{i\infty + \sigma} \overline{\phi_k}\, e^{pt}\, dp \quad . \tag{33}$$

The integration is along the straight line $Re(p) = \sigma > 0$. However, if we use expression (31) for $\overline{\phi_k}$, then this straight line can be moved far to the left half of the p-plane, looping around the poles of $\overline{\phi_k}$. Let p_1 denote the position of the pole that is closest to the imaginary axis. The path of integration with respect to this pole is shown in Figure 2. For large values of t only the residue at the pole will be of importance, the rest of the path contributing negligibly small amounts to the integral. In the neighborhood of $p = p_1$, $\overline{\phi_k}$ can be written as

$$\overline{\phi_k} = \frac{\eta(p)}{p - p_1} \tag{34}$$

where $\eta(p)$ is analytic. Substituting this expression into the inversion formula (33), we get

$$\phi_k(t) = 2\pi i \, \eta(p_1) \, e^{p_1 t} \quad . \tag{35}$$

Thus we see that for large values of t the time dependence of $\phi_k(t)$ is proportional to $e^{p_1 t}$.

Now p_1 must be found from equation (32). Landau determined p_1 for the two extreme cases of long waves (small k) and short waves (large k). Since p_1 is a function of k, he had to make an assumption about the behavior of p_1 as $k \to 0$ and as $k \to \infty$. He assumed that $Re(p_1) \to 0$ and $Im(p_1)$ remain finite as $k \to 0$. Hence in the case of long waves the pole is assumed to be at a finite distance from the imaginary axis and just below the real axis. The path of integration is now the real axis with a downward semi-circular indentation at the pole (Figure 3), the radius of the semi-circle being large compared with $Im(ip_1/k)$ but small compared with $Re(ip_1/k)$. The integral along the semi-circle is πi multiplied by the residue at the pole. Thus equation (32) becomes

$$\frac{\omega_p^2}{k^2} \int_{-\infty}^{\infty} \frac{f_o'(u)\, du}{u - ip/k} + \pi i \frac{\omega_p^2}{k^2} f_o' \left(\frac{ip}{k}\right) = 1 \; . \tag{36}$$

Although the integration is along the straight portion of the path, it is approximately correct to integrate along the entire real axis. Since k is small we may expand the integrand in powers of k and integrate term by term. That is,

$$\int_{-\infty}^{\infty} \frac{f_o'(u)\,du}{u - ip/k} = -\frac{k}{ip}\int_{-\infty}^{\infty} \frac{f_o'(u)\,du}{1 - \frac{ku}{ip}} = \tag{37}$$

$$-\frac{k}{ip}\int_{-\infty}^{\infty} \left[1 + \frac{ku}{ip} + \left(\frac{ku}{ip}\right)^2 + \left(\frac{ku}{ip}\right)^3 + \cdots\right] f_o'(u)\,du \quad .$$

Since $\int_{-\infty}^{\infty} f_o'(u)\,du = 0$, $\int_{-\infty}^{\infty} u\, f_o'(u)\,du = -1$,

$\int_{-\infty}^{\infty} u^2\, f_o'(u)\,du = 0$, and $\int_{-\infty}^{\infty} u^3\, f_o'(u)\,du = -3\left(\frac{\varkappa T}{m}\right)$,

the first four terms of expansion (37) yield (the higher terms are neglected)

$$\int_{-\infty}^{\infty} \frac{f'_o(u)\,du}{u - ip/k} = -\frac{k^2}{p^2} + 3\,\frac{\varkappa T}{m}\,\frac{k^4}{p^4} \quad . \tag{38}$$

With the aid of this evaluation equation (36) becomes

$$\frac{\omega_p^2}{k^2}\left(-\frac{k^2}{p^2} + \frac{3\varkappa T}{m}\,\frac{k^4}{p^4}\right) + \pi i\,\frac{\omega_p^2}{k^2}\, f_o'\!\left(\frac{ip}{k}\right) = 1 \quad . \tag{39}$$

It is interesting to note that if we neglect the contribution of the singularity we get

$$\frac{\omega_p^2}{k^2}\left(-\frac{k^2}{p^2}\right) + \frac{3\varkappa T}{m}\,\frac{k^4}{p^4}) = 1$$

from which it follows that

$$-p^2 = \omega_p^2 + \frac{3\mathcal{K}T}{m} k^2 \tag{40}$$

or

$$p = -i\omega = -i\omega_p \left(1 + \frac{3}{2}\frac{\mathcal{K}T}{m} k^2\right) \quad . \tag{41}$$

However, if we do not neglect the contribution from the singularity, p becomes complex. Let $p = -i\omega - \gamma$. Then by substituting into equation (39), ω and γ may be determined. Landau determined an approximate value of γ by neglecting the term $3\mathcal{K}T\, k^4/(mp^4)$ and by solving the resulting equation

$$-\frac{\omega_p^2}{p^2} + \pi i \frac{\omega_p^2}{k^2} f_o'\left(\frac{ip}{k}\right) = 1 \tag{42}$$

by successive approximations, and found it to be

$$\gamma = \omega_p \sqrt{\frac{\pi}{8}} \frac{1}{(k\rho)^3} e^{-\frac{1}{2(k\rho)^2}} \quad . \tag{43}$$

Thus Landau obtained the following expression for the time dependence of $\phi_k(t)$:

$$\exp\left\{-i\omega_p \left(1 + \frac{3}{2}\frac{\mathcal{K}T}{m} k^2\right) t - \gamma t\right\} \quad . \tag{44}$$

It differs from Vlasov's result by the damping factor $\exp(-\gamma t)$. The damping in this case of long waves is small. Landau also calculated the damping in the other limiting case of short wavelengths and found it to be very large.

A physical explanation* of this large damping is that those electrons whose velocities are near the singularity $u = ip/k$ are trapped in the potential wave. Due to the fact that $f_o(u)$ is a decreasing function of u, there will be more electrons on the slow side of the singularity than on the fast side. As the slow electrons are speeded up to the phase velocity, energy is extracted from the wave and hence the wave is attenuated. However, for large attenuations there is some question as to the

*Bohm, D. and E. P. Gross, Phys. Rev. **75**, 1851 (1949).

applicability of the linearization upon which this theory is based.

One of the assumptions upon which the above calculation rests is that $g_k(u)$ of equation (30) is an entire function of u. Whether or not this assumption is satisfied depends on the nature of the initial distribution $f(x,\underline{v},0)$ because $g_k(u)$ by definition is related to $f(x,\underline{v},0)$, viz.,

$$g_k(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x,\underline{v},0)\, e^{-ikx}\, dx\, dv_y\, dv_z \quad .$$

For $g_k(u)$ to have singularities in the finite part of the u-plane is not impossible. If $g_k(u)$ possesses singularities, then $\overline{\phi_k}$ has these singularities in addition to the ones produced by the zeros of the denominator, and in computing $\phi_k(t)$ one must take them into consideration. Thus we see that Landau's result is restricted to initial distributions that make $g_k(u)$ an entire function. But this restriction is not serious and can be removed in a straightforward manner.

Another assumption that Landau makes concerns the analyticity of the distribution function. He assumes that the distribution function is analytic up to infinite velocities, a condition that cannot be strictly satisfied. A possible method of modifying the theory to fit the case of speed-limited electrons has been given by Twiss.*

Eigenfunction Aspect of the Problem.

The transform method of handling the problem as given by Landau, Twiss and others** is completely satisfactory. However, there is an alternative approach to the problem which considers the behavior of the plasma as a superposition of stationary waves or eigenfunctions. This approach has the advantage of lending itself more directly to physical interpretation and for this reason is of considerable interest. Mathematically the reformulation of the problem as an eigenvalue problem leads

*Twiss, R. G., Phys. Rev., 88, 1392 (1952)

**See also F. Berz, Thesis, Univ. of London, 1955; Proc. Phys. Soc. London, 69-B, 953 (1956) to a singular integral equation of a type that has been handled in recent years by Russian mathematicians.* We shall not attempt a discussion of this aspect of the problem here; we reserve it for another note.

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*Muskhelishvili, N.I., Singular Integral Equations, Groningen, (1953).

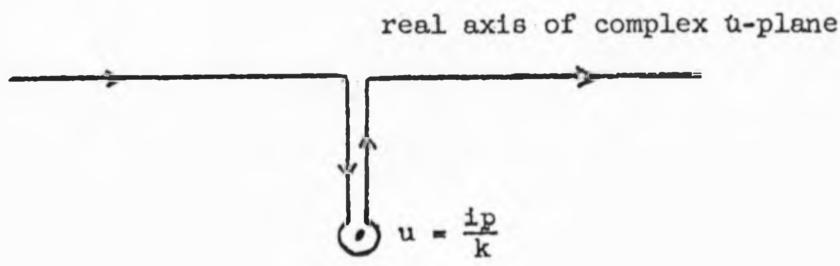

Fig. 1. Path of integration C in complex u-plane.

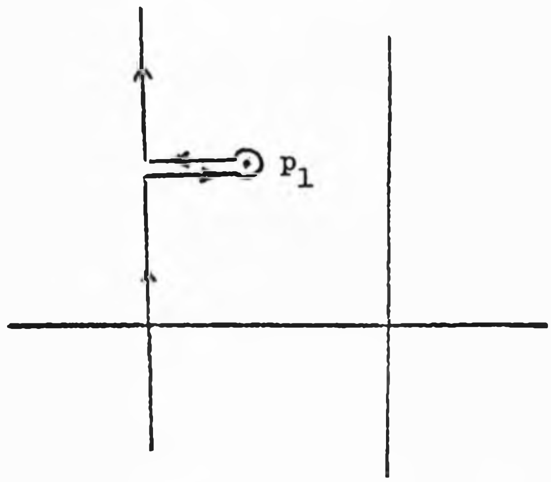

Fig. 2. Path of integration in p-plane with respect to pole $p = p_1$ of $\overline{\phi_k}$ closest to imaginary axis.

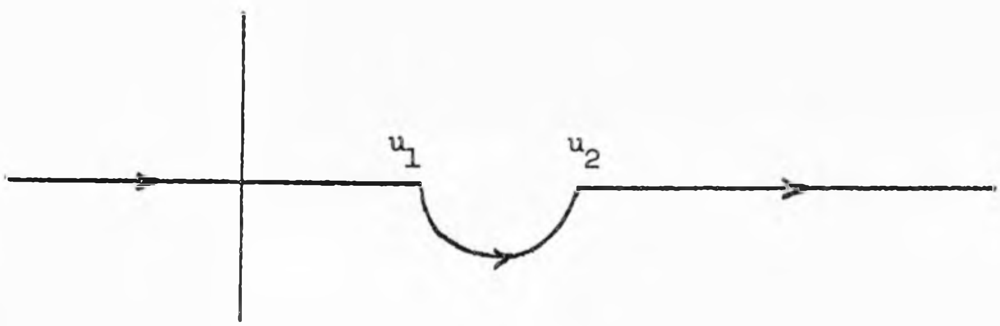

Fig. 3. Location of pole $u = ip_1/k$ for small k. Semicircle is chosen such that its radius is small compared with $\mathrm{Re}(ip_1/k)$ but large compared with $\mathrm{Im}(ip_1/k)$.